Triangulating Spaces

Definition Let $v_0, v_1, \ldots, v_k$ be points of the Euclidean n-space $\mathbb{R}^n$.The hyperplane spanned by these points consists of all linear combinations

$$\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_k v_k, \quad \lambda_i \in \mathbb{R} \text{ for each } 0 \leq i \leq k \text{ and the sum } \sum_{i=0}^{k} \lambda_i = 1.$$

The points are in general position if any subset of them spans a strictly smaller hyperplane. If we regard $\mathbb{R}^n$ as a vector space, then

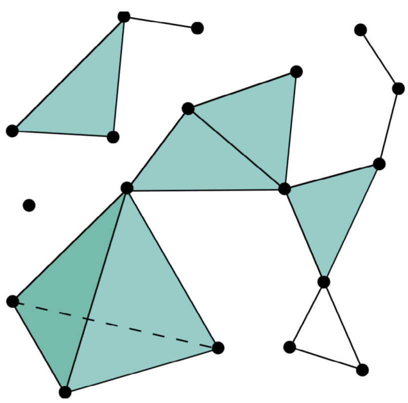
$$v_0, v_1, \ldots, v_k \text{ are in general position } \iff v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0 \quad \text{are linearly independent.}$$

Given $k+1$ points $v_0, v_1, \ldots, v_k$ in general position, we call the smallest convex set C (a polytope) containing them a simplex of dimension k (or a k-simplex). The points $v_0, v_1, \ldots, v_k$ are called the vertices of the simplex. It is eay to check that if C the smallest convex set containing $v_0, v_1, \ldots, v_k$, then

$$x \in C \iff x = \lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_k v_k \quad \text{where } \lambda_i \geq 0 \text{ for each } 0 \leq i \leq k \text{ and } \sum_{i=0}^{k} \lambda_i = 1.$$

Examples

- a 0-simplex is a point
- a 1-simplex is a closed line segment
- a 2-simplex is a triangle
- a 3-simplex is a tetrahedron (solid)

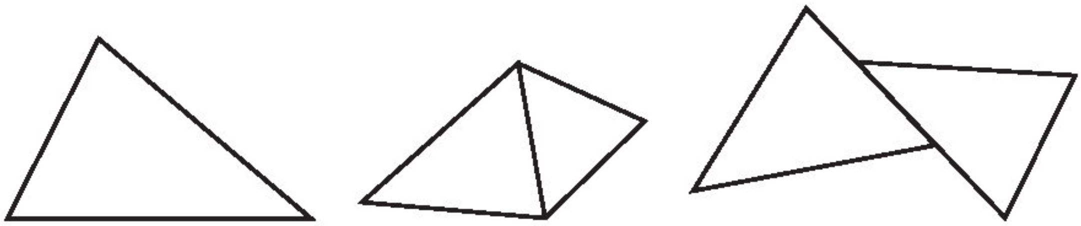


Definition Let A and B be simplices and let the vertices of B be a subset of the vertices of A. Then B is called a face of A, and it is denoted by $A < B$.

Two simplices are said to fit together nicely if they intersect only at a common face. A space is said to be triagulable if it is homeomorphic to the union of a finite collection of simplexes which fit together nicely in some Euclidean space.

Definition A simplicial complex K is a finite collection of simplices in some Euclidean space $\mathbb{R}^n$ that satisfies the following conditions:

1. Every face of a simplex from K is also in K.
2. The non-empty intersection of any two simplices A, B is a face of both A and B.



Definition A triangulation of a topological space X consists of a simplicial complex K and a homeomorphism $h : |K| \to X$, where $|K|$ is the union of the simplices of K, and it is called a polyhedron.

Definition A hole in a mathematical object is a topological structure which prevents the object from being continuously shrunk to a point. When dealing with topological spaces, a disconnectivity is interpreted as a hole in the space. Examples of holes are things like the "donut hole" in the center of the torus, and a domain removed from a plane.

The genus of a surface is defined as the largest topological invariant number of nonintersecting simple closed curves that can be drawn on the surface without separating it. Roughly speaking, it is the number of holes in a surface.

Polyhedral Formula A formula relating the number of polyhedron vertices V, faces F, and polyhedron edges E of a simply connected (i.e., genus 0) polyhedron (or polygon). It was discovered independently by Euler (1752) and Descartes, so it is also known as the Descartes-Euler polyhedral formula. The formula also holds for some, but not all, non-convex polyhedra.

The polyhedral formula states

$$V + F - E = 2,$$

where $V = N_0$ is the number of polyhedron vertices, $E = N_1$ is the number of polyhedron edges, and $F = N_2$ is the number of faces.

For genus g surfaces, the formula can be generalized to the Poincaré formula

$$\chi = V + F - E = \chi(g), \quad \text{where } \chi(g) = 2 - 2g$$

is the Euler characteristic, sometimes also known as the Euler-Poincaré characteristic. The polyhedral formula corresponds to the special case $g = 0$.

Definition A surface is closed if it is compact, connected, and has no boundary; in other words it is compact, connected, Hausdorff space in which each point has a neighborhood homeomorphic to the plane.

Classification Theorem Any closed surface is homeomorphic either to the sphere, or to the shpere with a finite number of handles added, or to the sphere with a finite number of discs removed and replaced by Möbius strips. No two of these surfaces are homeomorphic.

Gauss-Bonnet Formula Suppose M is a compact two-dimensional Riemannian manifold with boundary ∂M. Let K be the Gaussian curvature of M, and let k_g be the geodesic curvature of ∂M. Then

$$\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\pi\chi(M),$$

where dA is the element of area of the surface, and ds is the line element along the boundary of M.

If the boundary ∂M is piecewise smooth, then we interpret the integral $\int_{\partial M} k_g\, ds$ as the sum of the corresponding integrals along the smooth portions of the boundary, plus the sum of the angles by which the smooth portions turn at the corners of the boundary.

In particular,

- if T is a triangular region in M, then the Gauss-Bonnet formula is stated as

$$\int_T K\, dA = 2\pi\chi(T) - \sum_{i=1}^{3} \alpha_i - \int_{\partial T} k_g\, ds$$

  where $\sum_{i=1}^{3} \alpha_i$ is the sum of the (exterior) turning angle at vertex of T. So, if T is a geodesic triangle in M, i.e. T is a simply connected region bounded by geodesics in M, then the geodesic curvature $k_g = 0$ along the geodesics, the Euler characteristic $\chi(T) = 1$, and the Gauss-Bonnet formula becomes

$$\int_T K\, dA = 2\pi - \sum_{i=1}^{3} \alpha_i - \int_{\partial T} k_g\, ds = 2\pi - \sum_{i=1}^{3} \alpha_i$$

  Since the (exterior) turning angle at a corner is equal to π minus the interior angle, we can rephrase the Gauss-Bonnet formula as follows:

  The sum of interior angles of a geodesic triangle is equal to π plus the total curvature enclosed by the triangle:

$$\text{the sum of interior angles of } T = \sum_{i=1}^{3} (\pi - \alpha_i) = \pi + \int_T K\, dA \begin{cases} = \pi & \text{if } K \equiv 0 \text{ in } T \\ \geq \pi & \text{if } K \geq 0 \text{ in } T \\ \leq \pi & \text{if } K \leq 0 \text{ in } T \end{cases}$$

  In the case of the plane (where the Gaussian curvature is 0 and geodesics are straight lines), we recover the familiar formula for the sum of (interior) angles in an ordinary triangle. On the standard sphere, where the curvature is everywhere 1, we see that the (interior) angle sum of geodesic triangles is always bigger than π.

- if M is a closed Riemannian surface, then

$$\int_M K\, dA = 2\pi\chi(M).$$